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Complete (2,2) Bipartite Graphs

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Abstract

A bipartite graph G can be treated as a $(1,1)$ bipartite graph in the sense that, no two vertices in the same part are at distance one from each other. A $(2,2)$ bipartite graph is an extension of the above concept in which no two vertices in the same part are at distance two from each other. In this article, analogous to complete $(1,1)$ bipartite graphs which have the maximum number of pairs of vertices having distance one between them, a complete $(2,2)$ bipartite graph is defined as follows. A complete $(2,2)$ bipartite graph is a graph which is $(2,2)$ bipartite and has the maximum number of pairs of vertices (u,v) such that $d(u,v) = 2$. Such graphs are characterized and their properties are studied. The expressions are derived for the determinant, the permanent and spectral properties of some classes of complete $(2,2)$ bipartite graphs. A class of graphs among complete $(2,2)$ bipartite graphs having golden ratio in their spectrum is obtained.

Keywords: bipartite graphs; determinant; permanent; spectrum; golden ratio.

1 Introduction

The research background and motivation for the chosen topic are presented in the first part of this section, while the preliminary terminologies used in the article are presented in the second.

1.1 Motivation and Research Background

A bigraph or a bipartite graph is a graph G vertex set of which can be partitioned into two parts V_1 and V_2 such that no two vertices from the same part are at distance one. A considerable variation is taken from usual bipartite graphs and $(2,2)$ bipartite graphs are introduced in the literature by K. M. Prasad et. al as follows.

Definition 1.1. [7] *A graph G is said to be a $(2,2)$ bipartite graph if the vertex set $V(G)$ can be partitioned into a pair of nontrivial subsets V_1 and V_2 such that no two vertices from the same part are at distance two. A bipartition of $V(G)$ with the above properties is called a $(2,2)$ bipartition and the sets V_1 and V_2 are called parts of the $(2,2)$ bipartition.*

Throughout this article, a $(2,2)$ bipartite graph G is denoted with parts V_1, V_2 and $E(G) = E$ by $G(V_1 \cup V_2, E)$. Also, the usual bipartite graphs are referred as $(1,1)$ bipartite graphs. Trivially, every complete graph K_n and every totally disconnected graph is $(2,2)$ bipartite for every possible partition of its vertex set $V(G)$. The following is an example for $(2,2)$ bipartite graph showing the bipartition.

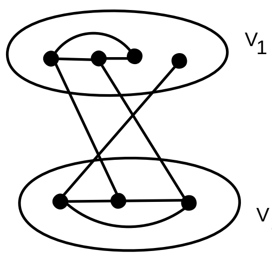

Figure 1: A $(2,2)$ bipartite graph.

It is interesting that every component in each of the parts of a $(2,2)$ bipartite graph is complete [7]. Further chracterization of $(2,2)$ bipartite graphs is given by the following theorem.

Theorem 1.1. [7] *The following statements are equivalent for every non trivial graph G.*

i) *G is $(2,2)$ bipartite.*

ii) *The vertex set V can be bipartitioned into V_1 and V_2 such that each component of the induced subgraphs $\langle V_1\rangle$ and $\langle V_2\rangle$ is complete and every vertex in $\langle V_i\rangle$ is adjacent with vertices of at most one component of $\langle V_j\rangle$, $j \neq i; i,j = 1,2$.*

The graphs $K_{1,3}$ and C_5 are not $(2,2)$ bipartite and any graph with $K_{1,3}$ or C_5 as an induced subgraph is also not $(2,2)$ bipartite [7]. A tree is $(2,2)$ bipartite if and only if it is a path [7]. Authors of [7] have also characterized graphs which are both $(1,1)$ bipartite and $(2,2)$ bipartite.

A complete $(1,1)$ bipartite graph is a $(1,1)$ bipartite graph which has the maximum number of pairs of vertices having distance one between them. Inspired by this insight of complete $(1,1)$ bipartite graphs, a complete $(2,2)$ bipartite graph is defined as follows.

Definition 1.2. *Let p, q be positive integers. A complete $(2,2)$ bipartite graph G with $|V_1| = p, |V_2| = q$ is a $(2,2)$ bipartite graph with maximum number of pairs of vertices (u,v) such that $d(u,v) = 2$.*

Before moving to the section of the main results of the article, some of the preliminary terminologies and notations used in the latter part of the article are provided.

1.2 Preliminaries

Matrices serve as models for graphs, illuminating their structure and allowing the use of simple yet powerful linear algebraic techniques to investigate them. The determinant, permanent, rank and eigenvalues are few of the powerful linear algebraic tools, which have been used extensively to study graphs. In specific, the parameters associated with the adjacency matrix of graphs are studied more extensively. For a graph G, the notations $rank(G)$, $det(G)$, $spec(G)$ and $per(G)$ describe the rank, determinant, eigenvalues and permanent of adjacency matrix of G respectively . If $\mu_1, \mu_2, \ldots, \mu_k$ are eigenvalues of the adjacency matrix of a graph G with multiplicities $m_1, m_2, \ldots, m_k$, respectively, then $spec(G)$ can be written as $spec(G) = \begin{pmatrix} \mu_1 & \mu_2 & \ldots & \mu_k \\ m_1 & m_2 & \ldots & m_k \end{pmatrix}$. A subgraph G_1 of a graph G is said to be elementary if every component of G_1 is a cycle or an edge. The following theorem gives the expressions for determinant and permanent of a graph in terms of its elementary spanning subgraphs [1].

Theorem 1.2. [1] *Let G be a graph on n vertices. Then,*

$$\det(G) = \sum_{G_1} (-1)^{n-k_1(G_1)-k_2(G_1)} 2^{k_2(G_1)}, \tag{1}$$

$$per(G) = \sum_{G_1} 2^{k_2(G_1)}, \tag{2}$$

where G_1 is the elementary spanning subgraph of G, $k_1(G_1)$ and $k_2(G_1)$ are the number of components in G_1 which are edges and cycles respectively.

Some more properties of determinants and permanents of graphs are discussed in [4]. Readers are referred to [10] for all the terminilogies used, but not described in this article.

This article comprises of four sections. Section 2 gives the characterization of complete $(2,2)$ bipartite graphs as well as some graph parameters associated. In Section 3, the results on $\det(G)$, $per(G)$ and $spectrum(G)$ of some cases of complete $(2,2)$ bipartite graphs are presented while Section 4 presents a class of graphs which are golden graphs. This article is concluded with an open problem for the readers.

2 Characterization

The following theorem characterizes complete $(2, 2)$ bipartite graphs.

Theorem 2.1. *Let G be a $(2, 2)$ bipartite graph with $(2, 2)$ bipartition $\{V_1, V_2\}$ where $|V_1| = p$ and $|V_2| = q$ such that $p + q = n$ and $p \geq q$. Then G is a complete $(2, 2)$ bipartite graph if and only if it satisfies both the conditions given below.*

(i) *V_1 induces the complete graph K_p.*

(ii) *Each vertex in V_2 is adjacent with exactly one vertex in V_1.*

Proof. Let G be a $(2, 2)$ bipartite graph and let v be a vertex in V_2. By characterization theorem of $(2, 2)$ bipartite graphs, each components of both the parts are complete and no vertex in any part is adjacent with vertices of more than one component of the other part. Hence, if the part V_1 has r components $C_1, C_2, \ldots, C_r$, then the number of pairs of vertices of the form (v, u_i) with $u_i \in V_1$ such that $d(v, u_i) = 2$ becomes maximum of $n_1 - 1, n_2 - 1, \ldots, n_r - 1$, where n_i is the number of vertices in C_i, $1 \leq i \leq r$. This becomes maximum when $r = 1$. When V_1 has only one component, the number of pairs of vertices (v, u_i) with $u_i \in V_1$ is maximum when the vertex v is adjacent with only one vertex of V_1. Thus, given $p \geq q$, the maximum number of vertices with distance two between them equal to 2 results when every vertex of V_2 is made adjacent with exactly one vertex of the only complete component of V_1. □

Remark 2.1. *For a given $(2, 2)$ bipartition $\{V_1, V_2\}$ with $|V_1| = p$ and $|V_2| = q$ $(p \geq q)$, the complete $(2, 2)$ bipartite graph has maximum number of pairs of vertices (u_i, v_k) with $d(u_i, v_k) = 2$ irrespective of the structure of $\langle V_2 \rangle$. The number of such pairs is given by $q\ (p - 1)$ if $p \geq q$.*

Following are some of the graphs (Figure 2) which are complete $(2, 2)$ bipartite with $|V_1| = 4$ and $|V_2| = 3$.

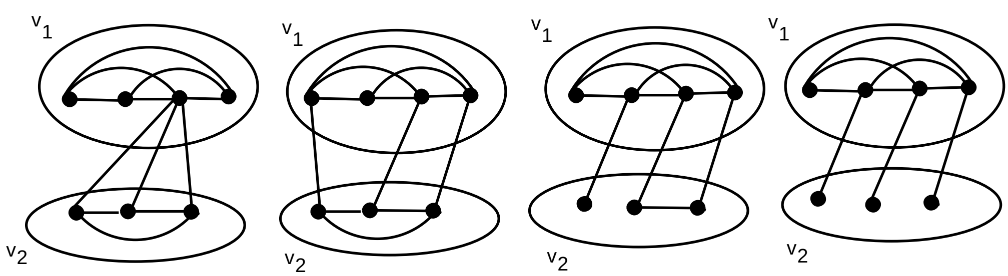

Figure 2: Complete $(2, 2)$ bipartite graphs on 7 vertices.

Note that there are 12 pairs of vertices (u, v) such that $d(u, v) = 2$, irrespetive of the structure of $\langle V_2 \rangle$.

For a given positive integer n, a $(1, 1)$ complete bipartite graph G of order n has maximum number of pairs of vertices (u, v) such that $d(u, v) = 1$ when the bipartition of G is $|V_1| = |V_2| = \frac{n}{2}$, when n is even and is $|V_1| = \frac{n+1}{2}, |V_2| = \frac{n-1}{2}$, when n is odd. Analogously, the following result gives the values of p and q such that a complete $(2, 2)$ bipartite graph $G(V_1 \cup V_2, E)$ with $|V_1| = p$ and $|V_2| = q$ has maximum number of pairs of vertices (u, v) such that $d(u, v) = 2$.

Corollary 2.1. *For a given positive integer n, a complete $(2,2)$ bipartite graph $G(V_1 \cup V_2, E)$ of order n has maximum number of pairs of vertices (u,v) such that $d(u,v) = 2$ if and only if*

$$|V_1| = |V_2| = \frac{n}{2} \text{ or } |V_1| = \frac{n}{2} + 1, |V_2| = \frac{n}{2} - 1, \quad for\ n\ even,$$

and

$$|V_1| = \frac{n+1}{2}, |V_2| = \frac{n-1}{2},\ for\ n\ \ odd.$$

Proof. The corollary above is proved separately when n is even and odd. Suppose $n = 2k$ for some integer k. Let $|V_1| = p, |V_2| = q$ such that $p + q = 2k$ and $p \geq q$. Let f be the number of pairs of vertices having distance two between them for given p and q. Since $q = 2k - p$, $f(p) = (2k - p)(p - 1) = 2kp - 2k - p^2 + p$. On maximizing f, $|V_1| = |V_2| = \frac{n}{2}$ or $|V_1| = \frac{n}{2} + 1$ and $|V_2| = \frac{n}{2} - 1$ are obtained.

Following the same procedure for the case when n is odd, the maxima is obtained when $|V_1| = \frac{n+1}{2}$ and $|V_2| = \frac{n-1}{2}$. □

For a complete $(2,2)$ bipartite graph $G(V_1 \cup V_2, E)$ with $|V_1| = p$ and $|V_2| = q$ $(p \geq q)$, we note the following.

Remark 2.2. *The bounds for the number of edges are given by,*

$$\frac{p^2 - p + 2q}{2} \leq |E(G))| \leq \frac{p^2 + q^2 - p + q}{2},\ if\ p > q,$$

and

$$\frac{p(p+1)}{2} \leq |E(G)| \leq p^2,\ if\ p = q.$$

The equalities are attained when V_2 induce $\overline{K_q}$ and K_q respectively.

Remark 2.3. *Observe that, the distance between any two vertices in G is $1, 2$ or 3. Hence, eccentricity of any vertex is $1, 2$ or 3. The diameter and radius of G are given by,*

$$diam(G) \leq 3,$$

$$rad(G) = \begin{cases} 1, & if\ both\ \langle V_i \rangle\ are\ complete\ and\ one\ of\ the\ vertices \\ & of\ V_1\ is\ adjacent\ with\ all\ the\ vertices\ of\ V_2, \\ 2, & else. \end{cases}$$

3 Further Results

Determinantal and permanental properties of adjacency matrices of graphs are some of the well studied areas. Note that if a graph G has a unique perfect matching, then $\det(G) = \pm 1$. Authors of [3] have proved that the determinant of the bipartite graph with at least two perfect matchings and with all cycle lengths divisible by four is zero. Also, the permanent of the biadjacency matrix of a bipartite graph enumerates the perfect matchings. In this section, some linear algebraic parameters of particular cases of complete $(2,2)$ bipartite graphs are explored. In a $(1,1)$ bipartite graph G with bipartition $V(G) = V_1 \cup V_2$, each vertex of V_i is adjacent to exactly one vertex of V_j $(i = 1, 2, i \neq j)$ results in a one factor graph. For the complete $(2,2)$ bipartite graph analogous to this, the following results are derived.

Theorem 3.1. *Let G be a complete $(2,2)$ bipartite graph on n (n is even) vertices such that $|V_1| = |V_2| = p = \frac{n}{2}$. Let both the parts V_i $(i = 1,2)$ induce the complete graph K_p and each vertex of V_i be adjacent to exactly one vertex of V_j for $i, j = 1, 2$ and $i \neq j$. Then the* $\det(G) = 0$.

Proof. After relabeling the vertices, the adjacency matrix A of the graph G can be viewed as

$$A = \left(\begin{array}{c|c} (J-I)_{p\times p} & (I)_{p\times p} \\ \hline (I)_{p\times p} & (J-I)_{p\times p} \end{array}\right),$$

where J is a square matrix of order p in which every entry is 1. Since I and $(J - I)$ commute,

$$\begin{aligned}
\det(A) &= \det\left[(J-I)^2 - I^2\right] \\
&= \det\left[(J-I)^2 - I\right] \\
&= \det\left[\begin{pmatrix} p-1 & p-2 & p-2 & \dots & p-2 \\ p-2 & p-1 & p-2 & \dots & p-2 \\ \vdots & & & & \\ p-2 & p-2 & p-2 & \dots & p-1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}\right] \\
&= \det\begin{pmatrix} p-2 & p-2 & p-2 & \dots & p-2 \\ p-2 & p-2 & p-2 & \dots & p-2 \\ \vdots & & & & \\ p-2 & p-2 & p-2 & \dots & p-2 \end{pmatrix} \\
&= 0.
\end{aligned}$$

□

The example for a graph that satisfies the conditions of the above theorem is given in Figure 3
.

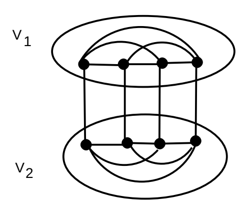

Figure 3: The complete $(2,2)$ bipartite graph analogous to one factor graph where $|V_1| = |V_2| = 4$.

Theorem 3.2. *Let G be a complete $(2,2)$ bipartite graph on n (n is even) vertices such that $|V_1| = |V_2| = p = \frac{n}{2}$. Let both the parts V_i $(i = 1,2)$ induce K_p and each vertex of V_i is adjacent to exactly one vertex of V_j for $i \neq j$ and $i, j = 1, 2$. Then,*

$$spec(G) = \begin{pmatrix} p & p-2 & 0 & -2 \\ 1 & 1 & p-2 & p-1 \end{pmatrix}.$$

Proof. The adjacency matrix A of the graph G can be written as (after relabeling the vertices)

$$A = \left(\begin{array}{c|c} (J-I)_{p\times p} & (I)_{p\times p} \\ \hline (I)_{p\times p} & (J-I)_{p\times p} \end{array}\right),$$

where J is a square matrix of order p.

It is known that, $eig\begin{pmatrix} M & N \\ N & M \end{pmatrix} = \{eig(M+N), eig(M-N)\}$, where $eig(A)$ represents eigenvalues of the matrix A. Thus, $eig(A) = \{eig(J_p), eig(J_p - 2I)\}$. Since $eig(J_p)$ are $p, 0$ with respective multiplicities $1, p-1$ and $eig(J_p - 2I)$ are $p-2, -2$ with respective multiplicities $1, p-1$, the result follows. □

The Corollary 3.1 follows from the above theorem.

Corollary 3.1. *Let G be a complete $(2, 2)$ bipartite graph on n (n is even) vertices such that $|V_1| = |V_2| = p = \frac{n}{2}$. Let both the parts V_i $(i = 1, 2)$ induce K_p and each vertex of V_i is adjacent to exactly one vertex of V_j for $i \neq j$ and $i, j = 1, 2$. Then $rank(G) = (p + 1)$.*

Proof. The proof follows from the fact that the graph G has $p + 1$ nonzero eigenvalues counting the multiplicities. □

Among all complete $(1, 1)$ bipartite graphs on n vertices, the star graph $K_{1,n-1}$ has maximum number of pairs (u, v) with $d(u, v) = 2$. The star graph is a special case of complete $(1, 1)$ bipartite graph where at least one of the parts has cardinality one. The star graph has $\det(K_{1,n-1}) = per(K_{1,n-1}) = 0$ and $spec(K_{1,n-1}) = \begin{pmatrix} \sqrt{n-1} & -\sqrt{n-1} & 0 \\ 1 & 1 & (n-2) \end{pmatrix}$. Analogous to star graphs, a complete $(2, 2)$ bipartite graph with at least one part has cardinality one (Figure 4) is considered.

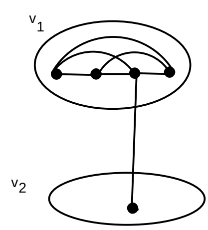

Figure 4: Complete $(2, 2)$ bipartite graph analogous to star graph.

The next theorem gives determinant, permanent and spectrum of such graphs.

Theorem 3.3. *Let G be a connected complete $(2, 2)$ bipartite graph such that $|V_1| = p > 1$ and $|V_2| = 1$. Let V_1 induces K_p and V_2 induces K_1. Then $\det(G) = 0$ if and only if $p = 2$. Further,*

$$per(G) = D_{p-1},$$
$$\det(G) = (-1)^{p-3}(p-2),$$

where $D_n = n! \sum_{i=2}^{n} \frac{(-1)^i}{i!}$.

Proof. Let v_i be the only vertex of V_2, which is adjacent with a vertex u_k of V_1. With every elementary spanning subgraph of K_{p-1} induced by the vertices of V_1 other than u_k, one can associate the edge (u_k, v_i) to get an elementary spanning subgraph of G. Conversely, from every elementary spanning subgraph of G which involves the edge (u_k, v_i), one can get an elementary spanning subgraph of K_{p-1}, by removing the edge the edge (u_k, v_i). This association is both one-one and onto.

Every elementary spanning subgraph of G and K_{p-1} differ only by a K_2. Hence the corresponding terms in the expression for permanent (Equation 2 of Theorem 1.2) remain the same, where as the the corresponding terms in the expression for determinant (Equation 1 of Theorem 1.2) are of opposite sign. Thus $per(G) = Per(K_{p-1}) = D_{p-1}$ and $\det(G) = -\det(K_{p-1}) = (-1)^{p-3}(p-2)$. $\square$

Theorem 3.4. *Let G be a connected complete $(2,2)$ bipartite graph such that $|V_1| = p$, $|V_2| = 1$ and $p > 1$. Let V_1 induces K_p and V_2 induces K_1. Then (-1) is an eigenvalue of G with multiplicity $p-2$. Further, the eigenvector corresponding to (-1) is $\begin{pmatrix} 0 & c_2 & c_3 & \dots & c_{p-1} & \sum\limits_{i=1}^{p-1} c_i & 0 \end{pmatrix}^T$ where $c_i (1 \le i \le p-1)$ are arbitrary constants.*

Proof. After relabeling the vertices, the adjacency matrix A of the graph G can be written as

$$A = \left(\begin{array}{c|c} (J-I)_{p\times p} & (C)_{p\times 1} \\ \hline (C^T)_{1\times p} & (0)_{1\times 1} \end{array}\right),$$

where J is a square matrix of order p with each entry one and $C^T = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \end{pmatrix}$.

Consider

$$\det(A-\mu I) = \det \begin{pmatrix} -\mu & 1 & 1 & \dots & 1 & 1 \\ 1 & -\mu & 1 & \dots & 1 & 0 \\ \vdots & & & & & \\ 1 & 1 & 1 & \dots & -\mu & 0 \\ 1 & 0 & 0 & \dots & 0 & -\mu \end{pmatrix},$$

where μ is an eigenvale of A. Applying the elementary row operations $R_i = R_i - R_{i+1}$ for $i = 2, 3, ..., p-1$ and $R_j = R_j$ for $j = 1, p, p+1$ would yield,

$$\det(A-\mu I) = \det \begin{pmatrix} -\mu & 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & -\mu-1 & 1+\mu & 0 & \dots & 0 & 0 \\ 0 & 0 & -\mu-1 & 1+\mu & \dots & 0 & 0 \\ \vdots & & & & & & \\ 1 & 1 & 1 & 1 & \dots & -\mu & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & -\mu \end{pmatrix}$$

$$= (1+\mu)^{(p-2)} \det \begin{pmatrix} -\mu & 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & -1 & 1 & \dots & 0 & 0 \\ \vdots & & & & & & \\ 1 & 1 & 1 & 1 & \dots & -\mu & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & -\mu \end{pmatrix}.$$

Thus, (-1) is an eigenvalue with the multiplicity $p-2$. Consider the system of equations $AX = \mu X$ where X is the eigenvector given by

$$X = \begin{pmatrix} x_1 & x_2 & \dots & x_{p+1} \end{pmatrix}^T.$$

When $\mu = -1$, we get,

$$\begin{aligned} x_1 + x_2 + ... + x_p + x_{p+1} &= 0 \\ x_1 + x_2 + ... + x_p &= 0 \\ x_1 + x_2 + ... + x_p &= 0 \\ &\vdots \\ x_1 + x_2 + ... + x_p &= 0 \\ x_1 + x_{p+1} &= 0. \end{aligned}$$

Since the algebraic multiplicity of (-1) is $p-2$, the dimension of the eigenspace of (-1) must be $(p-2)$. It is observed that $x_{p+1} = x_1 = 0$ and the remaining $p-1$ equations conclude that $x_2 = c_2, x_3 = c_3, ..., x_{p-1} = c_{p-1}$ and $x_p = -c_2 - c_3 - ... - c_{p-1}$. □

Remark 3.1. *The characteristic polynomial of a complete* $(2, 2)$ *bipartite graph such that* $|V_1| = p$, $|V_2| = 1$ $(p > 1)$ *and the parts* V_1, V_2 *induce* K_p, K_1 *respectively is given by*

$$(x+1)^{p-2}\left[x^3 - (p-2)x^2 - px + (p-2)\right].$$

Now, slight variation can be seen in the structure of above graphs. That is, the case where $|V_1| = p, |V_2| = q, (p \geq q)$ and $\langle V_1\rangle$ is K_p and $\langle V_2\rangle$ is $\overline{K}_q$ is considered.

Theorem 3.5. *Let* G *be a connected complete* $(2, 2)$ *bipartite graph such that* $|V_1| = p, |V_2| = q, (p \geq q)$ *and the part* V_1 *induces* K_p *and* V_2 *induces* $\overline{K}_q$*. Then*

$$\det(G) = \begin{cases} (-1)^p, & if\ p = q, \\ (-1)^{p-q-1}(p-q-1), & if\ p \neq q\ and\ q\ is\ even, \\ (-1)^{p-q-2}(p-q-2), & if\ p \neq q\ and\ q\ is\ odd, \end{cases}$$

$$per(G) = \begin{cases} D_{p-q}, & p \neq q, \\ 1, & p = q, \end{cases}$$

where $D_n = n! \sum\limits_{i=2}^{n} \frac{(-1)^i}{i!}$.

Proof. The proof is similar to the proof of Theorem 3.3. Let $p > q$. The elementary spanning subgraphs H of G are union qK_2s and H_1, where H_1 is elementary spanning subgraphs of K_{p-q}. Also, there exists a one to one correspondance between elementary spanning subgraphs of G and K_{p-q}. That is, for each elementary spanning subgraph H_1 of K_{p-q}, there exists an elementary spanning subgraph H of G which is given by $q\ K_2 \cup H$. For each H of G and corresponding H_1 of K_{p-q}, the terms in the summation of 1 are same except k_1. The term k_1 corresponding to H is $k_1^{'} + q$, where $k_1^{'}$ is the number of components of H_1 which are K_2s. Therefore,

$$\begin{aligned} \det(G) &= \sum_H (-1)^{n-k_1^{'}(H_1)-q-k_2(H)} 2^{k_2(H)} \\ &= (-1)^q \sum_H (-1)^{n-k_1^{'}(H_1)-k_2(H)} 2^{k_2(H)} \\ &= (-1)^q\ \det(K_{p-q}). \end{aligned}$$

Since $\det(K_n) = (-1)^{n-1}(n-1)$, the result follows. Similarly, $per(G) = per(K_{p-q}) = D_{p-q}$. When $p = q$, the graph has only one elementary spanning subgraph given by union of K_2s which are p in number. Hence $\det(G) = (-1)^p$ and $per(G) = 1$. □

4 Golden Graphs

The golden ratio, also known as divine ratio has fascinated western philosophers, mathematicians, scientists and almost all intellectuals in all fields of research. In literature, the numbers $\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}$ are referred as golden ratios. A graph G is said to be a pure golden graph if all the eigenvalues of G are golden ratios[8]. A graph G is a pure golden tree if and only if G is P_4 [8]. A graph G is said to be a golden graph, if at least one eigenvalue of G is golden ratio [8]. Some of the golden graphs have been characterized in [9, 6]. It is noted that the path graph P_n is golden graph if and only if $n = 5k - 1$ and the cycle graph C_n is golden graph if and only if $n = 5k$ for some positive integer k [9]. We know that a complete $(1, 1)$ bipartite is never a golden graph for any m and n. Unlike $(1, 1)$ complete bipartite graphs, some classes of complete $(2, 2)$ bipartite graphs are golden graphs. In the next theorem, a complete $(2, 2)$ bipartite graphs having the golden ratios $\frac{-1+\sqrt{5}}{2}$ and $\frac{-1-\sqrt{5}}{2}$ as eigenvalues are obtained.

Theorem 4.1. *Let G be a connected complete $(2, 2)$ bipartite graph such that $|V_1| = p, |V_2| = q, (p \geq q)$ and V_1 induces K_p. Let $\langle V_2 \rangle$ contains $\overline{K_r}$. Then G is a golden graph.*

Proof. After relabeling the vertices, the adjacency matrix A of the graph G can be viewed as

$$A = \left(\begin{array}{c|c} (J-I)_{p\times p} & U_{p\times q} \\ \hline U^T_{q\times p} & V_{q\times q} \end{array}\right),$$

where the matrices U and V are of the following forms:

$$U = \left(\begin{array}{c|c} M_{(p-r)\times(q-r)} & 0_{(p-r)\times r} \\ \hline 0_{r\times(q-r)} & I_{r\times r} \end{array}\right) \quad \text{and} \quad V = \left(\begin{array}{c|c} K_{(q-r)\times(q-r)} & 0_{(q-r)\times r} \\ \hline 0_{r\times(q-r)} & 0_{r\times r} \end{array}\right).$$

The matrices M, K are with arbitrary entries and J is matrix in which each entry is one. For convinience, let

$$(J-I)_{p\times p} = \left(\begin{array}{c|c} (J-I)_{(p-r)\times(q-r)} & J_{(p-r)\times r} \\ \hline J_{r\times(q-r)} & (J-I)_{r\times r} \end{array}\right).$$

Note that the order of the identity matrix I is such that the operations mentioned above are well defined. Consider $AX = \mu X$ where μ is an eigenvalue and let $X = \begin{bmatrix} X_1 & X_2 & X_3 & X_4 \end{bmatrix}^T$ be the eigenvector. The resulting system of equations is given by,

$$(J-I)X_1 + JX_2 + MX_3 = \mu X_1, \tag{3}$$
$$JX_1 + (J-I)X_2 + IX_4 = \mu X_2, \tag{4}$$
$$M^T X_1 + KX_3 = \mu X_3, \tag{5}$$
$$X_2 = \mu X_4. \tag{6}$$

Taking $X_1 = X_3 = 0$, (3) implies, $X_2 \in Nullspace(J)$.
From (4), $(J - I)X_2 + X_4 = \mu X_2$, which implies $X_4 = (\mu + 1)X_2$ (since $JX_2 = 0$).
Subsituting the expression for X_4 into (6) would give $(\mu^2+\mu-1)X_4 = 0$. Since $X_4 \neq 0$, $\mu^2+\mu-1 = 0$. This implies $\mu = \frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}$ and G is a golden graph. □

For the graph $G(V_1 \cup V_2, E)$ given in Theorem 4.1, we note the following.

Remark 4.1. *In the proof of the above theorem, note that* $X_2 \in Nullspace(J)$, *where* J *is of the order* $(p-r)\times r$, *the dimension of which is* $r-1$. *The dimension of the eigenspace corresponding to the eigenvalues* $\frac{-1+\sqrt{5}}{2}$ *and* $\frac{-1-\sqrt{5}}{2}$ *is at least* $(r-1)$ *as* $X_1 = X_3 = 0$ *and* $X_4 = \frac{1}{\lambda}X_2$. *Hence the algebraic multiplicities of* $\frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}$ *is at least* $r-1$.

Remark 4.2. *Suppose* $\langle V_2\rangle$ *is* $\overline{K_r} \cup C$, *where* C *is either* K_{q-r} *or union of two or more complete graphs, then* $\frac{-1+\sqrt{5}}{2}$ *and* $\frac{-1-\sqrt{5}}{2}$ *are eigenvalues with multiplicities at least* $(r-1)$ *irrespective of the structure of* C. *The existence and multiplicity is depending only on the number of isolated vertices in* $\langle V_2\rangle$.

All the three graphs in the Figure 5 have eigenvalues $\frac{-1+\sqrt{5}}{2}$ and $\frac{-1-\sqrt{5}}{2}$ with multiplicities at least one.

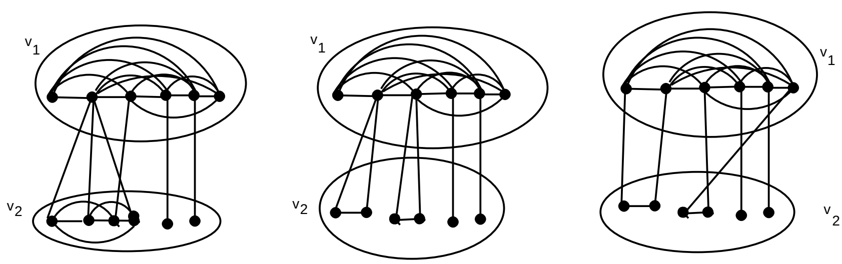

Figure 5: Complete $(2, 2)$ bipartite graphs on 12 vertices.

5 Conclusion

Friendship theorem is one of the famous theorems in Graph theory, which states that, in a group of people, if every two persons have a unique common friend, then there is a person in the group who is friend of everyone. The authors of [7] have modified the situation as follows: Given a collection of people, they can be partitioned into two groups, for any persons from the same group who are not friends of each other, there is no person who is a common friend. The above situation is modeled using $(2, 2)$ bipartite graphs. The $(2, 2)$ bipartite graph becomes a complete $(2, 2)$ bipartite graph when there are maximum number of pairs of people, each pair containing one person from each group who are not friends such that there is at least one person who is friend of both the persons in the pair. All the above concepts are expected to have some applications in social graph theory. The following open problem is proposed:

i) Characterize complete (k, k) bipartite graphs, where k is a positive integer greater than two.

ii) Decomposition of complete bipartie graphs ([2]) and complete k-partite graphs ([5]) have been discussed in the literature. Similar decomposition can be tried for complete $(2, 2)$ bipartite graphs.

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